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Kerr-Type Solution in Brans–Dicke Theory

Guang-Wen Ma[1]

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The Kerr-type solution in the Brans–Dicke theory should contain three parameters: a mass m, a rotational parameter a_0, and a coupling parameter ω. It goes over to the Kerr solution in Einstein's theory of general relativity in the limit $\omega \to \infty$. Using these conditions, we construct a special solution from Bruckman's solutions which can be regarded as a Kerr-type solution in the Brans–Dicke theory.

1. INTRODUCTION

The five-dimensional (5D) representation of gravitation was originally suggested by Kaluza and Klein in order to unify the gravitational and electric interactions (Kaluza, 1921; Klein, 1926). Many functions have been developed from it that mainly relate to the following three results: First, the Brans–Dicke theory (Brans and Dicke, 1961) can be equivalently expressed as a 5D theory by relating the fifth dimension with a scalar field (Belinsky and Khalatnikov, 1972). Second, through the Kaluza–Klein dimensional reduction procedure the 5D vacuum solutions are relevant to the construction of exact four-dimensional (4D) solutions with nonvanishing energy-momentum tensor (Belinsky and Ruffini, 1980), while the former can be found by using the inverse scattering method (ISM) under certain conditions (Belinsky and Zakharov, 1979; Ibanez and Verdaguer, 1986). Third, by identifying the quantity GM/c^2 (G is the Newtonian gravitational constant, M the rest mass of a particle, and c the speed of light) as the fifth coordinate, Wesson proposed a variable gravity, that is, 5D space-time-mass theory (Wesson, 1983, 1984), which can embody Mach's principle (Ma, 1990a,b).

Some years ago, Bruckman obtained stationary axially symmetric solutions in the 5D representation of the Brans–Dicke theory of gravitation using the ISM (Bruckman, 1986). Bruckman's work is very significant. However,

[1]Department of Physics, Zhengzhou University, Zhengzhou, Henan 450052, China.

we feel that these solutions are not vacuum solutions of the Brans–Dicke theory in the sense that the energy-momentum of the matter field is zero, and that their physical meaning is not fully brought to light. In the present article, we construct the Kerr-type solution in the Brans–Dicke theory on the basis of Bruckman's solutions, by choosing appropriate parameters.

2. VACUUM FIELD AND BRUCKMAN'S SOLUTIONS

According to Belinsky and Khalatnikov, the 5D line element has the form

$$-ds_{(5)}^2 = \gamma_{\mu\nu}(x^k)\, dx^\mu\, dx^\nu = A^2(\varphi)(dx^5)^2 + B^2(\varphi) g_{ij}(x^k)\, dx^i\, dx^j \qquad (1)$$

Here and in the following, Greek indices run from 1 to 5 and Latin indices from 1 to 4. Moreover, $\varphi = \varphi(x^k)$ is the scalar field in the Brans–Dicke theory and g_{ij} the 4D metric. The nonzero components of the 5D Ricci tensor are

$$\begin{aligned} R_{(5)55} &= \frac{A^2}{B^2}\left[\frac{A'}{A}\,\varphi_{;k}{}^{;k} + \frac{(A'B^2)'}{AB^2}\,\varphi_{;k}\varphi^{;k}\right] \\ R_{(5)ij} &= R_{ij} - \frac{(AB^2)'}{AB^2}\,\varphi_{;i;j} - \left(\frac{A''}{A} + 2\frac{B''}{B} - 2\frac{A'B'}{AB} - 4\frac{B'^2}{B^2}\right)\varphi_{;i}\varphi_{;j} \\ &\quad - \left[\frac{B'}{B}\,\varphi_{;k}{}^{;k} + \left(\frac{B''}{B} + \frac{B'^2}{B^2} + \frac{A'B'}{AB}\right)\varphi_{;k}\varphi^{;k}\right] g_{ij} \end{aligned} \qquad (2)$$

where R_{ij} is the 4D Ricci tensor, and all tensorial operations are computed with respect to g_{ij}. From the 5D field equation

$$R_{(5)\mu\nu} - \frac{1}{2} R_{(5)}\gamma_{\mu\nu} = -8\pi G T_{(5)\mu\nu}$$

one obtains

$$\begin{aligned} T_{(5)55} &= \frac{1}{8\pi G}\frac{A^2}{B^2}\left(\frac{1}{2}R + 3\frac{B'}{B}\,\varphi_{;k}{}^{;k} + 3\frac{B''}{B}\,\varphi_{;k}\varphi^{;k}\right) \\ T_{(5)5k} &= 0 \\ T_{(5)ij} &= -\frac{1}{8\pi G}\left[R_{ij} - \frac{1}{2}Rg_{ij} + \left(\frac{A'}{A} + 2\frac{B'}{B}\right)\varphi_{;i;j}\right. \\ &\quad + \left(\frac{A''}{A} + 2\frac{B''}{B} - 2\frac{A'B'}{AB} - 4\frac{B'^2}{B^2}\right)\varphi_{;i}\varphi_{;j} \\ &\quad \left. - \left(\frac{A'}{A} + 2\frac{B'}{B}\right)\varphi_{;k}{}^{;k} g_{ij} - \left(\frac{A''}{A} + 2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{A'B'}{AB}\right)\varphi_{;k}\varphi^{;k} g_{ij}\right] \end{aligned} \qquad (3)$$

The well-known Brans–Dicke field equations are

$$\varphi_{;k}{}^{;k} = \frac{8\pi}{3 + 2\omega} T_k^k \tag{4}$$

$$R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi}{\varphi} T_{ij} - \frac{\omega}{\varphi^2} \varphi_{;i}\varphi_{;j} + \frac{\omega}{2\varphi^2} \varphi_{;k}\varphi^{;k} g_{ij} - \frac{1}{\varphi} \varphi_{;i;j} + \frac{1}{\varphi} \varphi_{;k}{}^{;k} g_{ij}$$

From (3) and (4) one obtains that

$$T_{(5)55} = \frac{A^2}{8\pi G B^2} \left[\frac{8\pi}{3 + 2\omega} \left(\frac{\omega}{\varphi} + 3 \frac{B'}{B} \right) T_k^k - \left(\frac{\omega}{2\varphi^2} - 3 \frac{B''}{B} \right) \varphi_{;k}\varphi^{;k} \right]$$

$$\begin{aligned} T_{(5)ij} = \frac{1}{G\varphi} T_{ij} - \frac{1}{8\pi G} \Bigg[&\left(\frac{A'}{A} + 2 \frac{B'}{B} - \frac{1}{\varphi} \right) \varphi_{;i;j} \\ &+ \left(\frac{A''}{A} + 2 \frac{B''}{B} - 2 \frac{A'B'}{AB} - 4 \frac{B'^2}{B^2} - \frac{\omega}{\varphi^2} \right) \varphi_{;i}\varphi_{;j} \\ &- \left(\frac{A'}{A} + 2 \frac{B'}{B} - \frac{1}{\varphi} \right) \varphi_{;k}{}^{;k} g_{ij} \\ &- \left(\frac{A''}{A} + 2 \frac{B''}{B} - \frac{B'^2}{B^2} + \frac{A'B'}{AB} - \frac{\omega}{2\varphi^2} \right) \varphi_{;k}\varphi^{k;} g_{ij} \Bigg] \end{aligned} \tag{5}$$

If A and B are given by

$$A = \varphi^{[(3+2\omega)/3]^{1/2}}, \qquad B = \varphi^{\{1-[(3+2\omega)/3]^{1/2}\}/2} \tag{6}$$

we have

$$T_{(5)55} = \frac{1}{(3 + 2\omega) G\varphi} \frac{A^2}{B^2} \left(\omega + 3\varphi \frac{B'}{B} \right) T_k^k$$

$$T_{(5)ij} = \frac{1}{G\varphi} T_{ij} \tag{7}$$

Expressions (7) show that the 5D "vacuum" ($T_{(5)\mu\nu} = 0$) solutions correspond to the Brans–Dicke vacuum ($T_{ij} = 0$) field only if the relations (6) hold.

Furthermore, if the 5D metric $\gamma_{\mu\nu}(x^k)$ has the form

$$-ds_{(5)}^2 = f(\rho, z)\,(d\rho^2 + dz^2) + \gamma_{ab}(\rho, z)\, dx^a\, dx^b \tag{8}$$

with $a, b = 1', 2', 3'$ (here arabic numerals with primes are used in order to avoid confusion), one can solve the 5D vacuum field equations using the

ISM. Bruckman obtained the following set of new axially symmetric solutions with the Weyl–Levi-Civita solution as a "seed":

$$\begin{aligned}
\gamma_{1'1'} = \gamma_{\phi\phi} &= \frac{r^2}{\Omega}\left\{2\Omega + \frac{4\Gamma^2[\Gamma(r-\beta)/\beta + M]^2}{\Delta}\right.\\
&\qquad \left. - 4\,\frac{(b - a\cos\theta)^2}{\sin^2\theta} + a^2\sin^2\theta - \Delta\right\}\\
&\qquad \times\left(1 - \frac{2\beta}{r}\right)^{1-\delta-\bar{\nu}}\sin^2\theta\\
\gamma_{2'2'} = \gamma_{tt} &= -\frac{1}{\Omega}(\Delta - a^2\sin^2\theta)\left(1 - \frac{2\beta}{r}\right)^{\delta-\bar{\nu}} \qquad (9)\\
\gamma_{1'2'} = \gamma_{\phi t} &= -\frac{2\beta}{\Omega\Gamma}\left\{\Delta(a - b\cos\theta)\right.\\
&\qquad \left. - a\sin^2\theta\left[a^2 - b^2 - M^2 - \frac{\Gamma}{\beta}(r-\beta)M\right]\right\}\left(1 - \frac{2\beta}{r}\right)^{-\bar{\nu}}\\
\gamma_{3'3'} = \gamma_{55} &= (1 - 2\beta/r)^{2\bar{\nu}}\\
f &= \frac{\Omega}{(r-2\beta)^2}\,\frac{(1 - 2\beta/r)^{(\delta-1)(\delta-2)+3\bar{\nu}^2-\bar{\nu}}}{(1 - 2\beta/r + \beta^2/r^2\sin^2\theta)^{(\delta-1)^2+3\bar{\nu}^2}}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma &= \frac{1}{2}[\beta - (\beta^2 + a_0^2 - b_0^2)^{1/2}]\\
&\quad + \frac{1}{2}[\beta + (\beta^2 + a_0^2 - b_0^2)^{1/2}]\left(\frac{1 - 2\beta/r}{1 - 2\beta/r + \beta^2/r^2\sin^2\theta}\right)^{2\delta}\\
M &= \frac{1}{2}[-\beta + (\beta^2 + a_0^2 - b_0^2)^{1/2}]\\
&\quad + \frac{1}{2}[\beta^2 + (\beta^2 + a_0^2 - b_0^2)^{1/2}]\left(\frac{1 - 2\beta/r}{1 - 2\beta/r + \beta^2/r^2\sin^2\theta}\right)^{2\delta}\\
a &= \left(1 - \frac{2\beta}{r}\right)^{\delta}\left\{\frac{a_0 - b_0}{2}\left[1 - \frac{\beta}{r}(1 - \cos\theta)\right]^{-2\delta}\right. \qquad (10)\\
&\quad \left. + \frac{a_0 + b_0}{2}\left[1 - \frac{\beta}{r}(1 + \cos\theta)\right]^{-2\delta}\right\}
\end{aligned}$$

$$b = \left(1 - \frac{2\beta}{r}\right)^{\delta} \left\{ \frac{b_0 - a_0}{2} \left[1 - \frac{\beta}{r}(1 - \cos\theta)\right]^{-2\delta} + \frac{a_0 + b_0}{2} \left[1 - \frac{\beta}{r}(1 + \cos\theta)\right]^{-2\delta} \right\}$$

$$\Omega = \left[\frac{\Gamma}{\beta}(r - \beta) + M\right]^2 + (b - a\cos\theta)^2$$

$$\Delta = \frac{\Gamma^2}{\beta^2} r^2 \left(1 - \frac{2\beta}{r}\right)$$

and the spherical coordinates r, θ are related to the cylindrical coordinates ρ, z via

$$\rho = r(1 - 2\beta/r)^{1/2} \sin\theta$$
$$z = (r - \beta)\cos\theta \tag{11}$$

3. CONSTRUCTION OF THE KERR-TYPE SOLUTION FROM BRUCKMAN'S SOLUTIONS

As mentioned above, the Bruckman solutions (9) are not yet the vacuum solutions in the Brans–Dicke theory, although it is not difficult to obtain the latter from the former; all one has to do is a straightforward calculation from (1), (6), and (9). In view of the fact that the physical meaning of the Kerr solution is quite clear, we should like to find the Kerr-type solution in the Brans–Dicke theory from Bruckman's solutions. The key point is choosing the parameters δ, $\bar{\nu}$, β, a_0, and b_0. When $\delta = \bar{\nu} = 0$ and $b_0 = 0$ one can obtain the Kerr solution from (9), as Bruckman noticed, but such a solution is just one in the theory of general relativity because $\gamma_{3'3'}$, and thus φ, become constants in this case. We think that the Kerr-type solution in the Brans–Dicke theory should satisfy the following three conditions: (i) it should contain only three parameters, mass m, rotational parameter a_0, and coupling parameter ω; (ii) it goes to the Kerr solution in the theory of general relativity in the limit $\omega \to \infty$; (iii) it becomes the Schwarzschild-type solution in the absence of rotation. These conditions lead us to choose $b_0 = 0$ and

$$(\delta + 1)^2 + 3\bar{\nu}^2 = 1 \tag{12}$$

In such a case we have

$$\delta = (1 - 3\bar{\nu}^2)^{1/2} - 1 \tag{13}$$

where we have made the choice of (13) from two possible solutions of δ in order to obtain the Schwarzschild solution but not the trivial flat-space solution when $a_0 = 0$ and $\omega \to \infty$. The remaining tasks are to determine the relation of $\bar{\nu}$ with ω and that of β with the geometrical mass m. For our present purpose, let us consider the solution in the absence of rotation, which is

$$
\begin{aligned}
\gamma_{1'1'} &= \gamma^2(1 - 2\beta/r)^{[1-\bar{\nu}-(1-3\bar{\nu}^2)^{1/2}]}\sin^2\theta \\
\gamma_{2'2'} &= -(1 - 2\beta/r)^{(1-3\bar{\nu}^2)^{1/2}-\bar{\nu}} \\
\gamma_{1'2'} &= 0 \\
\gamma_{3'3'} &= (1 - 2\beta/r)^{2\bar{\nu}} \\
f &= (1 - 2\beta/r)^{[1-\bar{\nu}-(1-3\bar{\nu}^2)^{1/2}]}(1 - 2\beta/r + (\beta^2/r^2)\sin^2\theta)^{-1}
\end{aligned}
\tag{14}
$$

Transforming into a spherical coordinate system by using (11) and combining (1) with (6), we obtain

$$
\begin{aligned}
\varphi &= (1 - 2\beta/r)^{\bar{\nu}/[(3+2\omega)/3]^{1/2}} \\
g_{tt} &= -(1 - 2\beta/r)^{(1-3\bar{\nu}^2)^{1/2}-\bar{\nu}/[(3+2\omega)/3]^{1/2}} \\
g_{rr} &= (1 - 2\beta/r)^{-(1-3\bar{\nu}^2)^{1/2}-\bar{\nu}/[(3+2\omega)/3]^{1/2}} \\
g_{\theta\theta} &= r^2(1 - 2\beta/r)^{1-(1-3\bar{\nu}^2)^{1/2}-\bar{\nu}/[(3+2\omega)/3]^{1/2}} \\
g_{\phi\phi} &= g_{\theta\theta}\sin^2\theta
\end{aligned}
\tag{15}
$$

Using the transformation

$$
r = \bar{r}(1 + \beta/2\bar{r})^2 \tag{16}
$$

the coordinate system becomes isotropic, and the scalar field φ and the metric transform into

$$
\begin{aligned}
\varphi &= \left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{2\bar{\nu}/[(3+2\omega)/3]^{1/2}} \\
\bar{g}_{tt} &= -\left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{2\{(1-3\bar{\nu}^2)^{1/2}-\bar{\nu}/[(3+2\omega)/3]^{1/2}\}} \\
\bar{g}_{\bar{r}\bar{r}} &= (1 - \beta/2\bar{r})^{2\{1-(1-3\bar{\nu}^2)^{1/2}-\bar{\nu}/[(3+2\omega)/3]^{1/2}\}} \\
&\quad \times (1 + \beta/2\bar{r})^{2\{1+(1-3\bar{\nu}^2)^{1/2}+\bar{\nu}/[(3+2\omega)/3]^{1/2}\}} \\
\bar{g}_{\theta\theta} &= \bar{r}^2\bar{g}_{\bar{r}\bar{r}} \\
\bar{g}_{\phi\phi} &= \bar{r}^2\bar{g}_{\bar{r}\bar{r}}\sin^2\theta
\end{aligned}
\tag{17}
$$

The Robertson expansion of $\bar{g}_{ij}$ gives

$$\bar{g}_{tt} \equiv -\{1 - 2\alpha m/\bar{r} + 2\bar{\beta} m^2/\bar{r}^2 + \cdots\}$$
$$= -(1 - 4\{(1 - 3\bar{\nu}^2)^{1/2} - \bar{\nu}/[(3 + 2\omega)/3]^{1/2}\}\beta/2r$$
$$+ 8\{(1 - 3\bar{\nu}^2)^{1/2} - \bar{\nu}/[(3 + 2\omega)/3]^{1/2}\}^2\beta^2/4\bar{r}^2 + \cdots) \quad (18)$$
$$\bar{g}_{\bar{r}\bar{r}} \equiv \{1 + 2\bar{\gamma} m/\bar{r} + \cdots\}$$
$$= (1 + 4\{(1 - 3\bar{\nu}^2)^{1/2} + \bar{\nu}/[(3 + 2\omega)/3]^{1/2}\}\beta/2\bar{r} + \cdots)$$

The classical tests of the theory give the results $\alpha = \bar{\beta} = 1$ and $\bar{\gamma} = (\omega + 1)/(\omega + 2)$ (Weinberg, 1972); thus, from (17) and (18) we obtain

$$\{(1 - 3\bar{\nu}^2)^{1/2} - \bar{\nu}/[(3 + 2\omega)/3]^{1/2}\}\beta = m \quad (19)$$
$$\{(1 - 3\bar{\nu}^2)^{1/2} + \bar{\nu}/[(3 + 2\omega)/3]^{1/2}\}\beta = m(\omega + 1)/(\omega + 2)$$

It is easy to find the solution of (19), which is

$$\bar{\nu} = -1/[6(2 + \omega)]^{1/2}, \qquad \beta = [(3 + 2\omega)/(4 + 2\omega)]^{1/2} m \quad (20)$$

Finally, the set of special solutions which we seek is

$$\bar{g}_{tt} = -\frac{\Delta - a^2 \sin^2\theta}{\Omega}\left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{[2/(2+\omega)(3+2\omega)]^{1/2} + [2(3+2\omega)/(2+\omega)]^{1/2} - 2}$$

$$\bar{g}_{\bar{r}\bar{r}} = \frac{\Omega}{\bar{r}^2}\left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{[2/(2+\omega)(3+2\omega)]^{1/2} - 5[2(3+2\omega)/(2+\omega)]^{1/2} + 10}$$
$$\times \left[\left(1 - \frac{\beta^2}{4\bar{r}^2}\right)^2 + \frac{\beta^2}{\bar{r}^2}\sin^2\theta\right]^{2\{[2(3+2\omega)/(2+\omega)]^{1/2} - 2\}}$$

$$\bar{g}_{\theta\theta} = \bar{r}^2 \bar{g}_{\bar{r}\bar{r}}$$

$$\bar{g}_{\phi\phi} = \frac{\bar{r}^2}{\Omega}\left\{2\Omega + \frac{4\Gamma^2}{\beta^2\Delta}\left[\Gamma\bar{r}\left(1 + \frac{\beta^2}{4\bar{r}^2}\right) + \frac{M}{\beta}\right]^2\right. \quad (21)$$
$$\left. - \frac{4(b - a\cos\theta)^2}{\sin^2\theta} + a^2\sin^2\theta - \Delta\right\}$$
$$\times \left(1 + \frac{\beta}{2\bar{r}}\right)^4 \left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{[2/(2+\omega)(3+2\omega)]^{1/2} - [2(3+2\omega)/(2+\omega)]^{1/2} + 4} \sin^2\theta$$

$$\bar{g}_{\phi t} = -\frac{2\beta}{\Gamma\Omega}\left\{\Delta(a - b\cos\theta) - a\sin^2\theta\left[a^2 - b^2 - \frac{\Gamma}{\beta}\bar{r}\left(1 + \frac{\beta^2}{4\bar{r}^2}\right)M - M^2\right]\right\}$$
$$\times \left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{[2/(2+\omega)(3+2\omega)]^{1/2}}$$

where

$$\Gamma = \frac{\beta}{2}\left(\left\{1 - \left[1 + \left(\frac{a_0}{\beta}\right)^2\right]^{1/2}\right\} + \left\{1 + \left[1 + \left(\frac{a_0}{\beta}\right)^2\right]^{1/2}\right\}\right.$$

$$\left.\times \left[1 + \frac{\beta^2}{\bar{r}^2}\left(1 - \frac{\beta^2}{4\bar{r}^2}\right)^{-2} \sin^2\theta\right]^{2-[2(3+2\omega)/(2+\omega)]^{1/2}}\right)$$

$$M = \Gamma - \beta\left\{1 - \left[1 + \left(\frac{a_0}{\beta}\right)^2\right]^{1/2}\right\}$$

$$a = \frac{1}{2}\, a_0\left(1 + \frac{\beta}{2\bar{r}}\right)^{2\{[2(3+2\omega)/(2+\omega)]^{1/2}-2\}}\left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{[2(3+2\omega)/(2+\omega)]^{1/2}-2}$$

$$\times \left\{\left[1 - \frac{\beta}{2\bar{r}}\,(1 - 2\cos\theta)\right]^{2-[2(3+2\omega)/(2+\omega)]^{1/2}}\right.$$

$$\left. + \left[1 - \frac{\beta}{2\bar{r}}\,(1 + 2\cos\theta)\right]^{2-[2(3+2\omega)/(2+\omega)]^{1/2}}\right\} \tag{22}$$

$$b = a - a_0\left(1 + \frac{\beta}{2\bar{r}}\right)^{2\{[2(3+2\omega)/(2+\omega)]^{1/2}-2\}}$$

$$\times \left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^{[2(3+2\omega)/(2+\omega)]^{1/2}-2}$$

$$\times \left[1 - \frac{\beta}{2\bar{r}}\,(1 + 2\cos\theta)\right]^{2-[2(3+2\omega)/(2+\omega)]^{1/2}}$$

$$\Omega = \left[\frac{\Gamma}{\beta}\,\bar{r}\left(1 + \frac{\beta^2}{4\bar{r}^2}\right) + M\right]^2 + (b - a\cos\theta)^2$$

$$\Delta = \frac{\Gamma^2}{\beta^2}\,\bar{r}^2\left(\frac{1 - \beta/2\bar{r}}{1 + \beta/2\bar{r}}\right)^2$$

It can be verified that the solution (21) becomes the Boyer–Lindquist–Kerr solution in the limit $\omega \to \infty$ by use of the coordinate transformation

$$\tilde{t} = t + 2a_0\phi \tag{23}$$

$$\tilde{r} = \bar{r}(1 + \beta/2\bar{r})^2 - \beta + (\beta^2 + a_0^2)^{1/2}$$

We can say that such a special solution is the Kerr-type solution in the Brans–Dicke theory.

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